EIGENVALUES OF PERTURBED LAPLACE OPERATORS ON COMPACT MANIFOLDS

ASMA HASSANNEZHAD

Abstract. We obtain upper bounds for the eigenvalues of the Schrödinger operator $L = \Delta_g + q$ depending on integral quantities of the potential q and a conformal invariant called the *min-conformal volume*. Moreover, when the Schrödinger operator L is positive, integral quantities of q which appear in upper bounds, can be replaced by the mean value of the potential q. The upper bounds we obtain are compatible with the asymptotic behavior of the eigenvalues. We also obtain upper bounds for the eigenvalues of the weighted Laplacian or the Bakry–Émery Laplacian $\Delta_\phi = \Delta_g + \nabla_g \phi \cdot \nabla_g$ using two approaches: First, we use the fact that Δ_ϕ is unitarily equivalent to a Schrödinger operator and we get an upper bound in terms of the L^2-norm of $\nabla_g \phi$ and the min-conformal volume. Second, we use its variational characterization and we obtain upper bounds in terms of the L^∞-norm of $\nabla_g \phi$ and a new conformal invariant. The second approach leads to a Buser type upper bound and also gives upper bounds which do not depend on ϕ when the Bakry-Émery Ricci curvature is non-negative.

Contents

1. Introduction and statement of the results

In this paper, we study upper bound estimates for the eigenvalues of Schrödinger operators and weighted Laplace operators or Bakry–Émery Laplace operators.

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Schrödinger Operator. Let (M, g) be a compact Riemannian manifold of dimension m and $q \in C^0(M)$. The eigenvalues of the Schrödinger operator $L := \Delta_g + q$ acting on functions constitute a non-decreasing, semi-bounded sequence of real numbers going to infinity.

$$\lambda_1(\Delta_g + q) \leq \lambda_2(\Delta_g + q) \leq \cdots \leq \lambda_k(\Delta_g + q) \leq \cdots \nearrow \infty.$$

The well-known Weyl law which describes the the asymptotic behavior of the eigenvalues of the Laplacian [2] can be easily extended to the eigenvalues of Schrödinger operators on compact Riemannian manifolds:

$$\lim_{k\to\infty} \lambda_k(\Delta_g + q) \left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} = \alpha_m, \tag{1}$$

where $\alpha_m = 4\pi^2 \omega_m^{-\frac{2}{m}}$ and ω_m is the volume of the unit ball in $\mathbb{R}^m$.

It describes that normalized eigenvalues, $\lambda_k(\Delta_g + q)\left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}}$, asymptotically tend to a constant depending only on the dimension. However, upper bounds of normalized eigenvalues in general cannot be independent of geometric invariants and the potential q (see [5] or the introduction of [9]). We shall obtain upper bounds for normalized eigenvalues depending on some geometric invariants and integral quantities of the potential q. Moreover, these upper bounds are compatible with the asymptotic behavior in (1) i.e. they tend asymptotically to a constant depending only on the dimension as k goes to infinity.

Numerous articles are devoted to study how the eigenvalues of L can be controlled in terms of geometric invariants of the manifold and quantities depending on the potential. From the variational characterization of eigenvalues, it is easy to see that

$$\lambda_1(\Delta_g + q) \leq \frac{1}{\mu_g(M)} \int_M q d\mu_g.$$

For the second eigenvalue $\lambda_2(\Delta_g + q)$, an upper bound in terms of the mean value of the potential q and a conformal invariant was obtained by El Soufi and Ilias [7, Theorem 2.2]:

$$\lambda_2(\Delta_g + q) \leq m \left(\frac{V_c([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + \frac{\int_M q d\mu_g}{\mu_g(M)}, \tag{2}$$

where $V_c([g])$ is the conformal volume defined by Li and Yau [13] which only depends on the conformal class of g, denoted $[g]$.

For a compact orientable Riemannian surface (Σ_γ, g) of genus γ, they obtained the following inequality as a consequence of Inequality (2):

$$\lambda_2(\Delta_g + q) \leq \frac{8\pi}{\mu_g(\Sigma_\gamma)} \left[\frac{\gamma + 3}{2}\right] + \frac{\int_{\Sigma_\gamma} q d\mu_g}{\mu_g(\Sigma_\gamma)}, \tag{3}$$

where $[\frac{\gamma+3}{2}]$ is the integer part of $\frac{\gamma+3}{2}$.

For higher eigenvalues of Schrödinger operators, Grigor'yan, Netrusov and Yau [8] proved a general and abstract result that can be stated in the case of Schrödinger operators as follows: Given positive constants N and C_0, assume that a compact Riemannian manifold (M, g) has the $(2, N)$-covering property (i.e. each ball of radius r can be covered by N balls of radius $r/2$) and $\mu_g(B(x, r)) \leq C_0 r^2$ for every $x \in M$ and every $r > 0$. Then for every $q \in C^0(M)$ we have [8, Theorem 1.2 (1.14)]:

$$\lambda_k(\Delta_g + q) \leq \frac{Ck + \delta^{-1}\int_M q^+ d\mu_g - \delta \int_M q^- d\mu_g}{\mu_g(M)}, \tag{4}$$

where $\delta \in (0, 1)$ is a constant which depends only on N, $C > 0$ is a constant which depends on N and C_0, and $q^{\pm} = \max\{|\pm q|, 0\}$.
Moreover, if L is a positive operator [8, Theorem 5.15], then

$$\lambda_k(\Delta_g + q) \leq \frac{Ck + \int_M q d\mu_g}{\epsilon \mu_g(M)}, \tag{5}$$

where $\epsilon \in (0, 1)$ depends only on N and C depends on N and C_0.

The above inequalities in dimension 2 have special feature as follows. Let Σ_γ be a compact orientable Riemannian surface of genus γ. Then for every Riemannian metric g on Σ_γ and every $q \in C^0(\Sigma_\gamma)$ we have [8, Theorem 5.4]:

$$\lambda_k(\Delta_g + q) \leq \frac{Q(\gamma + 1)k + \delta^{-1}\int_{\Sigma_\gamma} q^+ d\mu_g - \delta \int_{\Sigma_\gamma} q^- d\mu_g}{\mu_g(\Sigma_\gamma)},$$

where $\delta \in (0, 1)$ and $Q > 0$ are absolute constants.

Inequalities (4) and (5) are not compatible with the asymptotic behavior regarding to the power of k, except in dimension 2. Yet, for surfaces, the limit of the above upper bound for normalized eigenvalues depends on the genus γ as k goes to infinity. Therefore, it is not compatible with (1).

We obtain upper bounds which generalize and improve the above inequalities without imposing any condition on the metric and which are compatible with the asymptotic behavior. Before stating our theorem, we need to recall the definition of the *min-conformal volume*. For a compact Riemannian manifold (M, g), its *min-conformal volume* is defined as follows [9].

$$V([g]) = \inf\{\mu_{g_0}(M) \;:\; g_0 \in [g],\ \mathrm{Ricci}_{g_0} \geq -(m-1)\}.$$

Theorem 1.1. *There exist positive constants $\alpha_m \in (0, 1)$, B_m and C_m depending only on m such that for every compact m-dimensional Riemannian manifold (M, g), every potential $q \in C^0(M)$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_g + q) \leq \frac{\alpha_m^{-1}\int_M q^+ d\mu_g - \alpha_m \int_M q^- d\mu_g}{\mu_g(M)} + B_m\left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m\left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}, \tag{6}$$

In particular, when the potential q is nonnegative one has

$$\lambda_k(\Delta_g + q) \leq A_m \frac{\int_M q d\mu_g}{\mu_g(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}, \qquad (7)$$

where $A_m = \alpha_m^{-1}$.

We also obtain upper bounds for eigenvalues of positive Schrödinger operators. Note that the positivity of the Schrödinger operator $L = \Delta_g + q$ implies that $\int_M q \geq 0$ and q here may not be nonnegative. The following upper bound generalizes Inequalities (5) and (7).

Theorem 1.2. *There exist constants $A_m > 1$, B_m and C_m depending only on m such that if $L = \Delta_g + q$, $q \in C^0(M)$ is a positive operator then for every compact m-dimensional Riemannian manifold (M^m, g) and every $k \in \mathbb{N}^*$ we have*

$$\lambda_k(\Delta_g + q) \leq A_m \frac{\int_M q d\mu_g}{\mu_g(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

Given the Schrödinger operator $L = \Delta_g + q$, for every $\varepsilon > 0$, the Schrödinger operator $\tilde{L} = \Delta_g + q - \lambda_1(L) + \varepsilon$ is positive and $\lambda_k(\tilde{L}) = \lambda_k(L) - \lambda_1(L) + \varepsilon$. When ε goes to zero, Theorem 1.1 leads to the following:

Corollary 1.1. *Under the assumptions of Theorem 1.1 we get*

$$\begin{aligned}\lambda_k(\Delta_g + q) \leq A_m \frac{\int_M q d\mu_g}{\mu_g(M)} &+ (1 - A_m)\lambda_1(\Delta_g + q) \\ &+ B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.\end{aligned}$$

In the 2-dimensional case, for a compact orientable Riemannian surface (Σ_γ, g) of genus γ, thanks to the uniformization and Gauss-Bonnet theorems, one has $V([g]) \leq 4\pi\gamma$. Therefore, in compact orientable Riemannian surfaces, one can replace the min-conformal volume by the topological invariant $4\pi\gamma$ in the above inequalities.

Corollary 1.2. *There exist absolute constants $a \in (0, 1)$, A and B such that, for every compact orientable Riemannian surface (Σ_γ, g) of genus γ, every potential $q \in C^0(M)$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_g + q)\mu_g(\Sigma_\gamma) \leq \int_{\Sigma_\gamma} (aq^+ - a^{-1}q^-) d\mu_g + A\gamma + Bk. \qquad (8)$$

And if L is a positive operator then

$$\lambda_k(\Delta_g + q)\mu_g(\Sigma_\gamma) \leq a \int_{\Sigma_\gamma} q d\mu_g + A\gamma + Bk.$$

An interesting application of Theorem 1.1 is the case of weighted Laplace operators or Bakry–Émery Laplace operators.

Bakry–Émery Laplacian. Let (M, g) be a Riemannian manifold and $\phi \in C^2(M)$. The corresponding weighted Laplace operator $\triangle_\phi$ is defined as follows.

$$\Delta_\phi = \Delta_g + \nabla_g \phi \cdot \nabla_g.$$

This operator is associated with the quadratic functional $\int_M |\nabla_g f|^2 e^{-\phi} d\mu_g$ i.e.

$$\int_M \Delta_\phi f h e^{-\phi} d\mu_g = \int_M \langle \nabla_g f, \nabla_g h \rangle e^{-\phi} d\mu_g.$$

This operator is an elliptic operator on $C_c^\infty(M) \subseteq L^2(e^{-\phi} d\mu_g)$ and can be extended to a self-adjoint operator with the weighted measure $e^{-\phi} d\mu_g$. In this sense, it arises as a generalization of the Laplacian. The weighted Laplace operator Δ_ϕ is also known as the diffusion operator or the Bakry–Émery Laplace operator which is used to study the diffusion process (see for instance, the pioneering work of Bakry and Émery [1], the paper of Lott [14], and Lott and Villani [15] on this topic). The triple (M, g, ϕ) is called a Bakry–Émery manifold where $\phi \in C^2(M)$ and (M, g) is a Riemannian manifold with the weighted measure $e^{-\phi} d\mu_g$ (see [16], [18]). The interplay between geometry of M and the behavior of ϕ is mostly taken into account by means of new notion of curvature called the Bakry–Émery Ricci tensor[1] that is defined as follows

$$\mathrm{Ricci}_\phi = \mathrm{Ricci}_g + \mathrm{Hess}\phi.$$

Our aim is to find upper bounds for the eigenvalues of Δ_ϕ denoted by $\lambda_k(\Delta_\phi)$ in terms of the geometry of M and of properties of ϕ.
Upper bounds for the first eigenvalue $\lambda_1(\Delta_\phi)$ of complete non-compact Riemannian manifolds have been recently considered in several works (see [17],[19],[20], [22] and [23]). These upper bounds depend on the L^∞-norm of $\nabla_g \phi$ and a lower bound of the Bakry–Émery Ricci tensor:
Let (M, g, ϕ) be a complete non-compact Bakry–Émery manifold of dimension m with $\mathrm{Ricci}_\phi \geq -\kappa^2(m-1)$ and $|\nabla_g \phi| \leq \sigma$ for some constants $\kappa \geq 0$ and $\sigma > 0$. Then we have [20, Proposition 2.1] (see also [17], [22] and [23]):

$$\lambda_1(\Delta_\phi) \leq \frac{1}{4}((m-1)\kappa + \sigma)^2. \tag{9}$$

In particular, if $\mathrm{Ricci}_\phi \geq 0$, then we have

$$\lambda_1(\Delta_\phi) \leq \frac{1}{4}\sigma^2. \tag{10}$$

We consider compact Bakry–Émery manifolds and we present two approaches to obtain upper bounds for the eigenvalues of the Bakry–Émery Laplace operator in terms of the geometry of M and of the properties of ϕ.

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[1] The Bakry–Émery Ricci tensor Ricci_ϕ is also referred to as the ∞-Bakry–Émery Ricci tensor. We will denote Ricci_ϕ and $\mathrm{Hess}\phi$ by $\mathrm{Ricci}_\phi(M, g)$ and $\mathrm{Hess}_g\phi$ wherever any confusion might occur.

First approach. One can see that Δ_ϕ is unitarily equivalent to the Schrödinger operator $L = \Delta_g + \frac{1}{2}\Delta_g\phi + \frac{1}{4}|\nabla_g\phi|^2$ (see for example [19, page 28]). Therefore, as a consequence of Theorem 1.2 we obtain an upper bound for $\lambda_k(\Delta_\phi)$ in terms of the min-conformal volume and the L^2-norm of $\nabla_g\phi$.

Theorem 1.3. *There exist constants A_m, B_m and C_m depending on $m \in \mathbb{N}^*$, such that for every m-dimensional compact Bakry–Émery manifold (M, g, ϕ), we have*

$$\lambda_k(\Delta_\phi) \leq A_m \frac{1}{\mu_g(M)} \|\nabla_g\phi\|^2_{L^2(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

It is worth noticing that in full generality, it is not possible to obtain upper bounds which do not depend on ϕ (see for instance [20, Section 2]). However, we will see that for compact manifolds with nonnegative Bakry–Émery Ricci curvature we can find upper bounds which do not depend on ϕ (see Corollary 1.4 below).
In the 2-dimensional case, as a result of Corollary 1.2 we obtain

Corollary 1.3. *There exist absolute constants $a \in (0, 1)$, A and B such that, for every compact orientable Riemannian surface (Σ_γ, g) of genus γ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_\phi)\mu_g(\Sigma_\gamma) \leq a\|\nabla_g\phi\|^2_{L^2(\Sigma_\gamma)} + A\gamma + Bk.$$

Second approch. It is based on using the technique introduced in [9] which was successfully applied for the Laplace operator Δ_g on Riemannian manifolds in [9, Theorem 1.1]. We obtain upper bounds for eigenvalues of Δ_ϕ in terms of a conformal invariant. We also obtain a Buser type upper bound for $\lambda_k(\Delta_\phi)$ (see below Corollary 1.5).

Definition 1.1. *Let (M, g, ϕ) be a compact Bakry–Émery manifold. We define the $\phi-$min conformal volume as*

$$V_\phi([g]) = \inf\{\mu_\phi(M, g_0) : g_0 \in [g], \mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)\}, \qquad (11)$$

where $\mu_\phi(M, g_0)$ is the weighted measure[2] *of M with respect to the metric g_0.*

Note that up to dilations[3] there is always a Riemannian metric $g_0 \in [g]$ such that $\mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)$. We are now ready to state our theorem.

[2]For a Bakry–Émery manifold (M, g, ϕ), when μ_ϕ is the weighted measure with respect to the metric g, we simply denote the weighted measure of a measurable subset A of M by $\mu_\phi(A)$ instead of $\mu_\phi(A, g)$.

[3]Notice Hessϕ and Ricci_g do not change under dilations. If $\mathrm{Ricci}_\phi(M, g) \geq -\kappa^2(m-1)g$, then $\forall\alpha > 0$, $\mathrm{Ricci}_\phi(M, g_0) := \mathrm{Ricci}_\phi(M, \alpha g) = \mathrm{Ricci}_\phi(M, g) \geq -\kappa^2(m-1)g = -\frac{\kappa^2}{\alpha}(m-1)g_0$.

Theorem 1.4. *There exist positive constants $A(m)$ and $B(m)$ depending only on $m \in \mathbb{N}^*$ such that for every compact Bakry–Émery manifold (M, g, ϕ) with $|\nabla_g \phi| \leq \sigma$ for some $\sigma \geq 0$ and for every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\} \left(\frac{V_\phi([g])}{\mu_\phi(M)}\right)^{\frac{2}{m}} + B(m) \left(\frac{k}{\mu_\phi(M)}\right)^{\frac{2}{m}}. \qquad (12)$$

If a metric g is conformally equivalent to a metric g_0 with $\mathrm{Ricci}_\phi(M, g_0) \geq 0$, then $V_\phi([g]) = 0$. Therefore, an immediate consequence of Theorem 1.4 is the following.

Corollary 1.4. *There exists a positive constant $A(m)$ which depends only on $m \in \mathbb{N}^*$ such that for every compact Bakry–Émery manifold (M, g, ϕ) with $V_\phi([g]) = 0$, and for every $k \in \mathbb{N}^*$*

$$\lambda_k(\Delta_\phi) \leq A(m) \left(\frac{k}{\mu_\phi(M)}\right)^{\frac{2}{m}}. \qquad (13)$$

The above upper bound is similar to the upper bound for the eigenvalues of the Laplacian in Riemannian manifolds (M, g) when $V([g]) = 0$ (see [11]).
If $\mathrm{Ricci}_\phi(M) > -\kappa^2(m-1)$ for some $\kappa \geq 0$, then for $g_0 = \kappa^2 g$ one has $\mathrm{Ricci}_\phi(M, g_0) > -(m-1)$ and $V_\phi([g]) \leq \mu_\phi(M, g_0) = \kappa^m \mu_\phi(M, g)$. Replacing in Inequality (12), we get a Buser type upper bound for the eigenvalues of the Bakry–Émery Laplacian.

Corollary 1.5 (Buser type upper bound)**.** *There are positive constants $A(m)$ and $B(m)$ depending only on $m \in \mathbb{N}^*$ such that for every compact Bakry–Émery manifold (M, g, ϕ) with $\mathrm{Ricci}_\phi(M) > -\kappa^2(m-1)$ and $|\nabla_g \phi| \leq \sigma$ for some $\kappa \geq 0$ and $\sigma \geq 0$, and for every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\} \kappa^2 + B(m) \left(\frac{k}{\mu_\phi(M)}\right)^{\frac{2}{m}}.$$

A weaker version of Corollary 1.5 can be proved directly by the classic idea used by Buser [3], Li and Yau [12]. We refer the reader to Appendix A where we give a simple direct proof.

Remark 1.1. *Notice that all of the results have been mentioned above for compact manifolds are also valid when bounded sudomains of complete manifolds with the Neumann boundary condition are considered.*

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2. Preliminaries and technical tools

We begin by recalling some definitions.
Basic definitions. A *capacitor* is a couple of Borel sets (F, G) in a topological space X such that $F \subsetneq G$.
We say that a metric space (X, d) satisfies the $(\kappa, N; \rho)$-*covering property* if each ball of radius $0 < r \leq \rho$ can be covered by N balls of radius $\frac{r}{\kappa}$. We sometimes call it local covering property when $\rho < \infty$.
For any $x \in X$ and $0 \leq r \leq R$, we define the annulus $A(x, r, R)$ as

$$A(x, r, R) := B(x, R) \setminus B(x, r) = \{y \in X : r \leq d(x, y) < R\}.$$

Note that $A(x, 0, R) = B(x, R)$. For any annulus $A(x, r, R)$ and $\lambda \geq 1$, set $\lambda A := A(x, \lambda^{-1} r, \lambda R)$. For $F \subseteq X$ and $r > 0$, we denote the *r-neighborhood of F* by F^r, that is

$$F^r = \{x \in X : d(x, F) \leq r\}.$$

Here, we state the key method that we use in order to obtain our results. This method was introduced in [9] and was inspired by two elaborate constructions given in [6] and [8]. It leads to construct a "nice" family of capacitors which is crucial to estimate the eigenvalues of Schrödinger operators and Bakry–Émery operators via capacities.

Capacity on Riemannian manifolds. For each capacitor (F, G) in a Riemannian manifold (M, g) of dimension m, we define the capacity and the m-capacity by:

$$\mathrm{cap}_g(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^2 d\mu_g, \quad \text{and} \quad \mathrm{cap}_{[g]}^{(m)}(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^m d\mu_g, \tag{14}$$

respectively, where $\mathcal{T} = \mathcal{T}(F, G)$ is the set of all functions $\varphi \in C_0^\infty(M)$ such that $\operatorname{supp} \varphi \subset G$, $0 \leq \phi \leq 1$ and $\varphi \equiv 1$ in a neighborhood of F. If $\mathcal{T}(F, G)$ is empty, then $\mathrm{cap}_g(F, G) = \mathrm{cap}_{[g]}^{(m)}(F, G) = +\infty$.

Proposition 2.1. ([10, Theorem 1.2.1], see also [9]) *Let (X, d, μ) be a metric measure space with a non-atomic Borel measure μ satisfying the $(2, N; \rho)$-covering property. Then for every $n \in \mathbb{N}^*$, there exists a family of capacitors $\mathcal{A} = \{(F_i, G_i)\}_{i=1}^n$ with the following properties:*

(i) $\mu(F_i) \geq \nu := \frac{\mu(X)}{8c^2 n}$, *where c is a constant depending only on N ;*
(ii) *the G_i's are mutually disjoint ;*
(iii) *the family $\mathcal{A}$ is such that either*
  (a) *all the F_i's are annuli and $G_i = 2F_i$, with outer radii smaller than ρ, or*

(b) *all the F_i's are domains in X and $G_i = F_i^{r_0}$, with $r_0 = \frac{\rho}{1600}$.*

The following lemma is a consequence of the above proposition.

Lemma 2.1. *Let (M^m, g, μ) be a compact Riemannian manifold with a non-atomic Borel measure μ. Then there exist positive constants $c(m) \in (0,1)$ and $\alpha(m)$ depending only on the dimension such that for every $k \in \mathbb{N}^*$ there exists a family $\{(F_i, G_i)\}_{i=1}^k$ of mutually disjoint capacitors with the following properties:*

(I) $\mu(F_i) > c(m)\frac{\mu(M)}{k}$;

(II) $\operatorname{cap}_g(F_i, G_i) \leq \frac{\mu_g(M)}{k}\left[\frac{1}{r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + \alpha(m)\left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}\right]$,

where $r_0 = \frac{1}{1600}$.

Proof of Lemma 2.1. Take the metric measure space (M, d_{g_0}, μ), where $g_0 \in [g]$ with $\text{Ricci}_{g_0} \geq -(m-1)$ and d_{g_0} is the distance associated to the Riemannian metric g_0. It is easy to verify that (M, d_{g_0}, μ) has the $(2, N; 1)$-covering property where N is a constant depending only on the dimension [9]. Therefore, Proposition 2.1 implies that for every $k \in \mathbb{N}^*$ there is a family of $3k$ mutually disjoint capacitors $\{(F_i, G_i)\}_{i=1}^{3k}$, satisfying the following properties.

(a) $\mu(F_i) > c(m)\frac{\mu(M)}{k}$, where $c(m) \in (0,1)$ is a positive constant depending only on the dimension ;
(b) all the F_i's are annuli, $G_i = 2F_i$ with outer radii smaller than 1 and $\operatorname{cap}_{[g]}^{(m)}(F_i, 2F_i) \leq Q_m$, where Q_m is a constant depending only on the dimension, or
(c) all the F_i's are domains in M and $G_i = F_i^{r_0}$ is the r_0-neighborhood of F_i, where $r_0 = \frac{1}{1600}$.

We refer the reader to [9, Proposition 3.1] for more details on the proof of the part (b). Hence, the family of $\{(F_i, G_i)\}_{i=1}^{3k}$ has the property (I).
We now show that at least k of them satisfy the property *(II)*. We first find an upper bound for the m-capacity $\operatorname{cap}_{[g]}^{(m)}(F_i, G_i)$. If all F_i's are annuli, we already have an estimate by the part (b). In the case (c), one can define a family of functions $\varphi_i \in \mathcal{T}(F_i, G_i)$, $1 \leq i \leq 3k$ so that $|\nabla_{g_0}\varphi_i| \leq \frac{1}{r_0}$. Then

$$\operatorname{cap}_{[g]}^{(m)}(F_i, G_i) \leq \int_M |\nabla_{g_0}\varphi_i|^m d\mu_{g_0} \leq \frac{1}{r_0^m}\mu_{g_0}(G_i).$$

Since $G_1, \ldots, G_{3k}$ are mutually disjoint, there exist at least $2k$ of them so that $\mu_{g_0}(G_i) \leq \mu_{g_0}(M)/k$. Similarly, there exist at least $2k$ sets (not necessarily the same ones) such that $\mu_g(G_i) \leq \mu_g(M)/k$. Therefore, up to re-ordering, we assume that the first k of them (i.e. $G_1, \ldots, G_k$) satisfy both of the two following inequalities

$$\mu_g(G_i) \leq \mu_g(M)/k, \qquad \mu_{g_0}(G_i) \leq \mu_{g_0}(M)/k.$$

Hence, in general, there exist k capacitors $(F_i, G_i), 1 \leq i \leq k$ with

$$\operatorname{cap}^{(m)}_{[g]}(F_i, G_i) \leq Q_m + \frac{1}{r_0^m}\frac{\mu_{g_0}(M)}{k}.$$

The left hand-side of the above inequality is a conformal invariant. Now, taking infimum over $g_0 \in [g]$ with $\mathrm{Ricci}_{g_0} \geq -(m-1)$ we get

$$\operatorname{cap}^{(m)}_{[g]}(F_i, G_i) \leq Q_m + \frac{1}{r_0^m}\frac{V([g])}{k}.$$

Now, for every $\varepsilon > 0$, we consider plateau functions $\{f_i\}_{i=1}^k$, $f_i \in \mathcal{T}(F_i, G_i)$ with

$$\int_M |\nabla_g f_i|^m d\mu_g \leq \operatorname{cap}^{(m)}_{[g]}(F_i, G_i) + \varepsilon.$$

Therefore,

$$\begin{aligned}
\operatorname{cap}_g(F_i, G_i) &\leq \int_M |\nabla_g f_i|^2 d\mu_g \leq \left(\int_M |\nabla_g f_i|^m d\mu_g\right)^{\frac{2}{m}} \left(\int_M 1_{\operatorname{supp} f_i} d\mu_g\right)^{1-\frac{2}{m}} \\
&\leq \left(\operatorname{cap}^{(m)}_{[g]}(F_i, G_i) + \varepsilon\right)^{\frac{2}{m}} \mu_g(G_i)^{1-\frac{2}{m}} \\
&\leq \left(Q_m + \frac{1}{r_0^m}\frac{V([g])}{k} + \varepsilon\right)^{\frac{2}{m}} \mu_g(G_i)^{1-\frac{2}{m}} \\
&\leq \left[Q_m^{\frac{2}{m}} + \frac{1}{r_0^2}\left(\frac{V([g])}{k}\right)^{\frac{2}{m}} + \varepsilon^{\frac{2}{m}}\right] \left(\frac{\mu_g(M)}{k}\right)^{1-\frac{2}{m}}. \qquad (15)
\end{aligned}$$

where Inequality (15) is due to the well-know fact that

$$(a+b)^s \leq a^s + b^s$$

when a, b are nonnegative real numbers and $0 < s \leq 1$. Letting ε tend to zero, we obtain the property *(II)*. It completes the proof. $\square$

Capacity on Bakry–Émery manifolds. In an analogous way, we define the capacity in a Bakry–Émery manifold (M, g, ϕ). For each capacitor (F, G) in a Bakry–Émery manifold (M, g, ϕ) of dimension m, the capacity and the m-capacity is defined as:

$$\operatorname{cap}_\phi(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^2 d\mu_\phi, \quad \text{and} \quad \operatorname{cap}^{(m)}_\phi(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^m d\mu_\phi, \qquad (16)$$

respectively, where $\mathcal{T} = \mathcal{T}(F, G)$ is the set of all functions $\varphi \in C_0^\infty(M)$ such that $\operatorname{supp} \varphi \subset G$, $0 \leq \phi \leq 1$ and $\varphi \equiv 1$ in a neighborhood of F. If $\mathcal{T}(F, G)$ is empty, then $\operatorname{cap}_\phi(F, G) = \operatorname{cap}^{(m)}_\phi(F, G) = +\infty$.

We shall prove a similar lemma as Lemma 2.1. We start by showing that every compact Bakry–Émery manifold satisfies the assumptions of Proposition 2.1. Thanks to volume comparison theorem proved by Wei and Wylie [21] for Bakry–Émery manifolds, one can show that Bakry–Émery manifolds have local covering property (see Lemma 2.2 below).

Theorem 2.1 (*Volume comparison theorem*[21])**.** *Let (M, g, ϕ) be a compact Bakry–Émery manifold with $Ricci_\phi \geq \alpha(m-1)$. If $\partial_r \phi \geq -\sigma$, with respect to geodesic polar coordinates centered at x, then for every $0 < r \leq R$ we have (assume $R \leq \pi/2\sqrt{\alpha}$ if $\alpha > 0$)*

$$\frac{\mu_\phi(B(x,R))}{\mu_\phi(B(x,r))} \leq e^{\sigma R} \frac{v(m,R,\alpha)}{v(m,r,\alpha)}, \tag{17}$$

and in particular, letting r tend to zero yields

$$\mu_\phi(B(x,R)) \leq e^{\sigma R} v(m,R,\alpha), \tag{18}$$

where $v(m,r,\alpha)$ is the volume of a ball of radius r in the simply connected space form of constant sectional curvature α.

Lemma 2.2. *Let (M, g, ϕ) be a compact Bakry–Émery manifold with $Ricci_\phi \geq -\kappa^2(m-1)$ and $|\nabla_g \phi| \leq \sigma$ for some $\kappa \geq 0$ and $\sigma \geq 0$. There exist constants $N(m) \in \mathbb{N}^*$ and $\xi = \xi(\sigma, \kappa) > 0$ such that (M, g, ϕ) satisfies the $(2, N; \xi)$-covering property. Moreover, there exists a positive constant $C(m)$ such that for every $0 \leq r < R \leq \xi$ and $x \in M$, the annulus $A = A(x, r, R)$ satisfies $\mathrm{cap}_\phi^{(m)}(A, 2A)) \leq C(m)$.*

Proof. Take $\xi = min\{\frac{1}{\sigma}, \frac{1}{\kappa}\}$ (take $\xi = \infty$ if $\sigma = \kappa = 0$). We first show that (M, μ_ϕ) has the doubling property for $r < 4\xi$, i.e.

$$\mu_\phi(B(x,r)) \leq c\mu_\phi(B(x,r/2)), \quad 0 < r < 4\xi,$$

for some positive constant c. From this, it is easy to deduce that (M, μ_ϕ) has the $(2, N; \xi)$-covering property for example with $N = c^4$. To prove the doubling property, according to Inequality (17) we have

$$\frac{\mu_\phi(B(x,r))}{\mu_\phi(B(x,r/2))} \leq e^{\sigma r} \frac{v(m,r,-\kappa^2)}{v(m,r/2,-\kappa^2)} = e^{\sigma r} \frac{v(m,\kappa r,-1)}{v(m,\kappa r/2,-1)}.$$

Take $\tilde{r} := \kappa r$ and $\tilde{R} := \kappa R$. Hence, for every $0 < r < 4\xi = 4\min\{\frac{1}{\sigma}, \frac{1}{\kappa}\}$, we get

$$\begin{aligned} e^{\sigma r} \frac{v(m,\kappa r,-1)}{v(m,\kappa r/2,-1)} &\leq e^4 \frac{v(m,\tilde{r},-1)}{v(m,\tilde{r}/2,-1)}; \quad 0 < \tilde{r} < 4, \\ &\leq \sup_{\tilde{r} \in (0,4)} e^4 \frac{v(m,\tilde{r},-1)}{v(m,\tilde{r}/2,-1)} =: c(m). \end{aligned}$$

Thus,

$$\frac{\mu_\phi(B(x,r))}{\mu_\phi(B(x,r/2))} \leq c(m), \quad \text{for every } 0 < r < \xi.$$

Therefore, (M, g, ϕ) has $(2, N; \xi)$-covering property where $N = c^4(m)$.
To estimate the capacity of an annulus, we now follow the same argument

as in [9, page 3430]. Let $A = A(x, r, R)$ and let $f \in \mathcal{T}(A, 2A)$ be

$$f(y) = \begin{cases} 1 & \text{if} \quad y \in A(x,r,R) \\ \frac{2d_{g_0}(y, B(x, r/2))}{r} & \text{if} \quad y \in A(x, r/2, r) \text{ and } r \neq 0 \\ 1 - \frac{d_{g_0}(y, B(x,R))}{R} & \text{if} \quad y \in A(x, R, 2R) \\ 0 & \text{if} \quad y \in M \setminus A(x, r/2, 2R) \end{cases} . \tag{19}$$

We have

$$|\nabla_{g_0} f| \leq \frac{2}{r}, \quad \text{on } B(x,r) \setminus B(x, r/2),$$

$$|\nabla_{g_0} f| \leq \frac{1}{R}, \quad \text{on } B(x, 2R) \setminus B(x, R).$$

Therefore,

$$\begin{aligned} \operatorname{cap}_\phi^{(m)}(A, 2A) &\leq \int_M |\nabla_g f|^m d\mu_\phi \leq \big(\frac{2}{r}\big)^m \mu_\phi(A(x, r/2, r)) + \big(\frac{1}{R}\big)^m \mu_\phi(A(x, R, 2R)) \\ &\leq \big(\frac{2}{r}\big)^m \mu_\phi(B(x,r)) + \big(\frac{1}{R}\big)^m \mu_\phi(B(x, 2R)). \end{aligned}$$

Having Inequality (18), one gets

$$\begin{aligned} \operatorname{cap}_\phi^{(m)}(A, 2A) &\leq \left(\frac{2}{r}\right)^m e^{\sigma r} v(m, r, -\kappa^2) + \left(\frac{1}{R}\right)^m e^{2\sigma R} v(m, 2R, -\kappa^2) \\ &= \left(\frac{2}{\kappa r}\right)^m e^{\sigma r} v(m, \kappa r, -1) + \left(\frac{1}{\kappa R}\right)^m e^{2\sigma R} v(m, 2\kappa R, -1). \end{aligned}$$

Take $\tilde{r} := \kappa r$ and $\tilde{R} := \kappa R$. Hence, for every $0 < r < R \leq 2\xi = 2\min\{\frac{1}{\sigma}, \frac{1}{\kappa}\}$, we get

$$\begin{aligned} \operatorname{cap}_\phi^{(m)}(A, 2A) &\leq \left(\frac{2}{\tilde{r}}\right)^m e^2 v(m, \tilde{r}, -1) + \left(\frac{1}{\tilde{R}}\right)^m e^4 v(m, 2\tilde{R}, -1) \\ &\leq \sup_{\tilde{r}, \tilde{R} \in (0,2)} \left[\left(\frac{2}{\tilde{r}}\right)^m e^2 v(m, \tilde{r}, -1) + \left(\frac{1}{\tilde{R}}\right)^m e^4 v(m, 2\tilde{R}, -1)\right] \\ &=: C(m). \end{aligned} \tag{20}$$

This completes the proof. $\square$

Lemma 2.3. *Let (M^m, g, ϕ) be a compact Bakry–Émery manifold with $|\nabla_g \phi| \leq \sigma$ for some $\sigma \geq 0$. Then there exist positive constants $c(m) \in (0,1)$ and $\alpha(m)$ depending only on the dimension such that for every $k \in \mathbb{N}^*$ there exists a family $\{(F_i, G_i)\}_{i=1}^k$ of capacitors with the following properties:*

(I) $\mu_\phi(F_i) > c(m)\frac{\mu_\phi(M)}{k}$,

(II) $\operatorname{cap}_\phi(F_i, G_i) \leq \frac{\mu_\phi(M)}{k}\left[\frac{1}{r_0^2}\left(\frac{V_\phi([g])}{\mu_\phi(M)}\right)^{\frac{2}{m}} + \alpha(m)\left(\frac{k}{\mu_\phi(M)}\right)^{\frac{2}{m}}\right]$,

where $\frac{1}{r_0} = 1600\max\{\sigma, 1\}$.

Proof. We consider the Bakry–Émery manifold (M, g, ϕ) as the metric measure space (M, d_{g_0}, μ_ϕ) where $g_0 \in [g]$ with $\mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)$ and μ_ϕ is the weighted measure with respect to the metric g. According to Lemma 2.2, this space has the $(2, N, \xi)$-covering property with $\xi = \min\{\frac{1}{\sigma}, 1\}$. Having Proposition 2.1 and Lemma 2.2, and following steps analogous to those in Lemma 2.1, implies that for every $k \in \mathbb{N}^*$, there exists a family of k mutually disjoint capacitors $\{F_i, G_i\}$ satisfying the following properties.

(a) $\mu_\phi(F_i) \geq c(m)\frac{\mu_\phi(M)}{k}$, where $c(m) \in (0, 1)$ is a positive constant depending only on the dimension, and $\mu_\phi(G_i) \leq \frac{\mu_\phi(M)}{k}$.
(b) all the F_i's are annuli, $G_i = 2F_i$ with outer radii smaller than ξ and $\mathrm{cap}_\phi^{(m)}(F_i, G_i) \leq C(m)$, where $C(m)$ is a constant defined in (20). or
(c) all the F_i's are domains in M, $G_i = F_i^{r_0}$ is the r_0-neighborhood of F_i and $\mathrm{cap}_\phi^{(m)}(F_i, G_i) \leq \frac{1}{r_0^2}\frac{V_\phi([g])}{k}$, with $r_0 = \frac{\xi}{1600}$

Hence, $\mathrm{cap}_\phi^{(m)}(F_i, G_i) \leq C(m) + \frac{1}{r_0^2}\frac{V_\phi([g])}{k}$. Now, for every $\varepsilon > 0$, we consider a family of functions $\{f_i\}_{i=1}^k$, $f_i \in \mathcal{T}(F_i, G_i)$ such that

$$\int_M |\nabla_g f_i|^m e^{-\phi} d\mu_g \leq \mathrm{cap}_\phi^{(m)}(F_i, G_i) + \varepsilon.$$

We repeat the same argument as before.

$$\begin{aligned}
\mathrm{cap}_\phi(F_i, G_i) &\leq \int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g \\
&\leq \left(\int_M |\nabla_g f_i|^m e^{-\phi} d\mu_g\right)^{\frac{2}{m}} \left(\int_M 1_{\mathrm{supp} f_i} e^{-\phi} d\mu_g\right)^{1-\frac{2}{m}} \\
&\leq \left[C(m)^{\frac{2}{m}} + \frac{1}{r_0^2}\left(\frac{V_\phi([g])}{k}\right)^{\frac{2}{m}} + \varepsilon^{\frac{2}{m}}\right] \left(\frac{\mu_\phi(M)}{k}\right)^{1-\frac{2}{m}}.
\end{aligned}$$

Having $\frac{1}{r_0} = \frac{1600}{\xi} = 1600 \max\{\sigma, 1\}$ and letting ε tend to zero, we obtain the property (II). It completes the proof. $\square$

3. Eigenvalues of Schrödinger operators

In this section, we prove Theorems 1.1 and 1.2. The idea of the proof is to construct a suitable family of test functions to be used in the variational characterization of the eigenvalues. Due to the min-max Theorem, we have the following variational characterization for the eigenvalues of the Schrödinger operator $L = \Delta_g + q$:

$$\lambda_k(\Delta_g + q) = \min_{V_k} \max_{0 \neq f \in V_k} \frac{\int_M |\nabla_g f|^2 d\mu_g + \int_M f^2 q d\mu_g}{\int_M f^2 d\mu_g},$$

where V_k is a k-dimensional linear subspace of $H^1(M)$ and μ_g is the Riemannian measure corresponding to the metric g.

According to this variational formula, for every family $\{f_i\}_{1=1}^k$ of disjointly supported test functions one has

$$\lambda_k(\Delta_g+q)\le \max_{i\in\{1,\dots,k\}}\frac{\int_M|\nabla_g f_i|^2d\mu_g+\int_M f_i^2 q d\mu_g}{\int_M f_i^2 d\mu_g}. \tag{21}$$

The potential $q\in C^0(M)$ is a signed function (notice that we can assume $q\in L^1(M)$ as well). We define a signed measure σ associated to the potential q by

$$\sigma(A)=\int_A q d\mu_g, \quad \text{for every measurable subset } A \text{ of } X.$$

For any signed measure ν we write $\nu=\nu^+-\nu^-$, where ν^+ and ν^- are the positive and negative parts of ν, respectively. For any signed measure ν and $0\le\delta\le 1$ we define a new signed measure ν_δ as $\nu_\delta:=\delta\nu^+-\nu^-$.
Let μ and ν be two signed measures on M. Then, according to [8, Lemma 4.3], the following inequality is satisfied.

$$(\mu+\nu)_\delta\ge\mu_\delta+\nu_\delta. \tag{22}$$

Proof of Theorem 1.1. For a real number $\lambda\in\mathbb{R}$ define $\mu_\lambda:=(\lambda\mu_g-\sigma)^+$ as a non-atomic Borel measure on M. We apply Lemma 2.1 to (M,g,μ_λ). Thus, for every $k\in\mathbb{N}^*$ and every $\lambda\in\mathbb{R}$, there exists a family $\{(F_i,G_i)\}_{i=1}^{2k}$ of $2k$ capacitors satisfying the properties (I) and (II) of Lemma 2.1.
From now on, we take $\lambda:=\lambda_k=\lambda_k(L)$. The property (I) yields

$$(\lambda_k\mu_g-\sigma)^+(F_i)\ge c(m)\frac{(\lambda_k\mu_g-\sigma)^+(M)}{2k}.$$

The measure $(\lambda_k\mu_g-\sigma)^-$ is also a non-atomic. Since G_i's are mutually disjoint, up to reordering, the first k of them satisfy

$$(\lambda_k\mu_g-\sigma)^-(G_i)\le\frac{(\lambda_k\mu_g-\sigma)^-(M)}{k}, \quad i\in\{1,\dots,k\}.$$

Therefore

$$\begin{aligned}(\lambda_k\mu_g-\sigma)^-(G_i)-(\lambda_k\mu_g-\sigma)^+(F_i)\le&\frac{(\lambda_k\mu_g-\sigma)^-(M)}{k}\\ &-c(m)\frac{(\lambda_k\mu_g-\sigma)^+(M)}{2k}.\end{aligned} \tag{23}$$

For every $\epsilon>0$ and every $1\le i\le k$, we choose $f_i\in\mathcal{T}(F_i,G_i)$ such that:

$$\int_M|\nabla_g f_i|^2d\mu_g\le \operatorname{cap}_g(F_i,G_i)+\epsilon. \tag{24}$$

Inequality (21) implies that there exists $i\in\{1,\cdots,k\}$ so that

$$\lambda_k\int_M f_i^2d\mu_g\le\int_M|\nabla_g f_i|^2d\mu_g+\int_M f_i^2qd\mu_g.$$

Hence, having Lemma 2.1 and Inequality (23) we get

$$\begin{aligned}
0 &\leq \int_M |\nabla_g f_i|^2 d\mu_g - \int_M f_i^2(\lambda_k - q) d\mu_g \\
&\leq \mathrm{cap}_g(F_i, G_i) + \epsilon - \int_M f_i^2(\lambda_k - q) d\mu_g \\
&\leq \frac{\mu_g(M)}{2k}\left[\frac{1}{r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + \alpha(m)\left(\frac{2k}{\mu_g(M)}\right)^{\frac{2}{m}}\right] + \epsilon \\
&\quad + \int_M f_i^2(\lambda_k - q)^- d\mu_g - \int_M f_i^2(\lambda_k - q)^+ d\mu_g \\
&\leq \frac{\mu_g(M)}{2k}\left[\frac{1}{r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + \alpha(m)\left(\frac{2k}{\mu_g(M)}\right)^{\frac{2}{m}}\right] + \epsilon \\
&\quad + \frac{(\lambda_k\mu_g - \sigma)^-(M)}{k} - c(m)\frac{(\lambda_k\mu_g - \sigma)^+(M)}{2k}. \qquad (25)
\end{aligned}$$

We now estimate the last two terms of the above inequality considering two alternatives:

Case 1. If $\lambda_k = \lambda_k(L)$ is positive, then applying Inequality (22) for the measure $\lambda_k\mu_g$ and signed measure $-\sigma$ with $\delta = \frac{c(m)}{2}$, we get

$$\begin{aligned}
\frac{c(m)}{2}(\lambda_k\mu_g - \sigma)^+(M) - (\lambda_k\mu_g - \sigma)^-(M) &\geq \frac{c(m)}{2}\sigma^-(M) - \sigma^+(M) \\
&\quad + \frac{c(m)}{2}\lambda_k\mu_g(M). \qquad (26)
\end{aligned}$$

Replacing (26) in (25), and letting ϵ tend to zero gives the following

$$\lambda_k \leq \frac{\frac{2}{c(m)}\sigma^+(M) - \sigma^-(M)}{\mu_g(M)} + \frac{1}{c(m)r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + \frac{\alpha(m)}{c(m)}\left(\frac{2k}{\mu_g(M)}\right)^{\frac{2}{m}}. \qquad (27)$$

Case 2. If $\lambda_k = \lambda_k(L)$ is non-positive, then applying Inequality (22) for the signed measures $\lambda_k\mu_g$ and $-\sigma$ with $\delta = \frac{c(m)}{2}$, implies

$$\begin{aligned}
\frac{c(m)}{2}(\lambda_k\mu_g - \sigma)^+(M) - (\lambda_k\mu_g - \sigma)^-(M) &\geq \frac{c(m)}{2}\sigma^-(M) - \sigma^+(M) \\
&\quad + \lambda_k\mu_g(M). \qquad (28)
\end{aligned}$$

Replacing (28) in (25) and letting ϵ go to zero gives the following

$$\lambda_k \leq \frac{\sigma^+(M) - \frac{c(m)}{2}\sigma^-(M)}{\mu_g(M)} + \frac{1}{2r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + \frac{\alpha(m)}{2}\left(\frac{2k}{\mu_g(M)}\right)^{\frac{2}{m}}. \qquad (29)$$

Therefore, $\lambda_k(L)$ is smaller than the sum of the right-hand sides of Inequalities (27) and (29). We finally obtain Inequality (6) with, for example, $\alpha_m = \frac{c(m)}{4}$. $\square$

Proof of Theorem 1.2 . We partly follow the spirit of the proof of [8, Theorem 5.15]. Take the measure metric space (M, g, μ_g). By Lemma 2.1, for every $k \in N^*$ there is a family of $2k$ disjoint capacitors $\{(F_i, G_i)\}_{i=1}^{2k}$ that satisfies the properties (I) and (II). For every $\varepsilon > 0$, let $\{f_i\}_{i=1}^{2k}$ be a family of test functions with $2f_i \in \mathcal{T}(F_i, G_i)$ and $4\int_M |\nabla_g f_i|^2 d\mu_g \leq \text{cap}_g(F_i, G_i) + \varepsilon$. We claim that this family satisfies the following property:

$$\sum_{i=1}^{2k} \int_M f_i^2 q d\mu_g \leq \sum_{i=1}^{2k} \int_M |\nabla_g f_i|^2 d\mu_g + \int_M q d\mu_g. \tag{30}$$

If we have Inequality (30) then

$$\begin{aligned}\sum_{i=1}^{2k} \int_M \left(|\nabla_g f_i|^2 + f_i^2 q\right) d\mu_g &\leq 2\sum_{i=1}^{2k} \int_M |\nabla_g f_i|^2 d\mu_g + \int_M q d\mu_g \\ &\leq k \max_i \text{cap}_g(F_i, G_i) + k\varepsilon + \int_M q d\mu_g.\end{aligned}$$

By the assumption $\int_M \left(|\nabla_g f_i|^2 + f_i^2 q\right) d\mu_g$ is positive for each $1 \leq i \leq 2k$. Therefore, at least k of them satisfy the following inequality (up to reordering we assume that the first k of them satisfy the inequality):

$$\int_M (|\nabla_g f_i|^2 + f_i^2 q) d\mu_g \leq \max_i \text{cap}_g(F_i, G_i) + \varepsilon + \frac{\int_M q d\mu_g}{k}. \tag{31}$$

Inequality (31) together with the bounds of $\text{cap}_g(F_i, G_i)$ and $\mu_g(F_i)$ given in Lemma 2.1, properties (I) and (II) lead to

$$\begin{aligned}\lambda_k(L) &\leq \max_i \frac{\int_M |\nabla_g f_i|^2 d\mu_g + \int_M f_i^2 q d\mu_g}{\int_M f_i^2 d\mu_g} \leq \frac{\max_i \text{cap}_g(F_i, G_i) + \varepsilon + \frac{1}{k}\int_M q d\mu_g}{\mu_g(F_i)} \\ &\leq \frac{1}{c(m) r_0^2} \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + \alpha(m) \left(\frac{2k}{\mu_g(M)}\right)^{\frac{2}{m}} + \frac{2k\varepsilon}{c(m)\mu_g(M)} + \frac{2\int_M q d\mu_g}{c(m)\mu_g(M)}.\end{aligned}$$

Hence, we get the desired inequality as ε tends to zero. It remains to prove Inequality (30) which is proved in [8, Section 5]; however, for the reader's convenience we repeat the proof. We define the function h by the following identity

$$\sum_{i=1}^{2k} f_i^2 + h^2 = 1. \tag{32}$$

Since $f_1, \dots, f_{2k}$ are disjointly supported and $0 \leq f_i \leq \frac{1}{2}$, hence, $h \geq \frac{1}{2}$. We now estimate the left-hand side of Inequality (30).

$$\int_M \left(\sum_{i=1}^{2k} f_i^2 + h^2 - h^2\right) q d\mu_g = \int_M q d\mu_g - \int_M h^2 q d\mu_g \leq \int_M q d\mu_g + \int_M |\nabla h|^2 d\mu_g, \tag{33}$$

where the last inequality comes from the fact that the Schrödinger operator L is positive. Identity (32) implies

$$-2h\nabla_g h = -\nabla_g h^2 = \sum_{i=1}^{2k} \nabla_g f_i^2 = 2\sum_{i=1}^{2k} f_i \nabla_g f_i.$$

Therefore,

$$|\nabla_g h|^2 \leq |2h\nabla_g h|^2 = \sum_{i=1}^{2k} |\nabla_g f_i^2|^2 = 4\sum_{i=1}^{2k} |f_i\nabla_g f_i|^2 \leq \sum_{i=1}^{2k} |\nabla_g f_i|^2. \qquad (34)$$

Combining Inequalities (33) and (34) we get Inequality (30). □

4. Eigenvalues of Bakry–Émery Laplace operators

In this section we consider eigenvalues of the Bakry–Émery Laplace operator Δ_ϕ on a Bakry–Émery manifold (M, g, ϕ), where M is a compact m-dimensional Riemannian manifold and $\phi \in C^2(M)$. We denote the weighted measure on M by μ_ϕ with

$$\mu_\phi(A) = \int_A e^{-\phi} d\mu_g, \quad \text{for every Borel subset } A \text{ of } M.$$

Proof of Theorem 1.3. As we mentioned in the introduction, one can see that $\Delta_\phi = \Delta_g + \nabla_g\phi \cdot \nabla_g$ is unitarily equivalent to the positive Schrödinger operator $L = \Delta_g + \frac{1}{2}\Delta_g\phi + \frac{1}{4}|\nabla_g\phi|^2$. Therefore, Theorem 1.2 yields

$$\begin{aligned}\lambda_k(\Delta_\phi) \quad &\leq \quad A_m \frac{1}{\mu_g(M)} \int_M \left(\frac{1}{2}\Delta_g\phi + \frac{1}{4}|\nabla_g\phi|^2\right) d\mu_g \\ &\quad + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.\end{aligned}$$

Stokes theorem implies that $\int_M \Delta_g\phi d\mu_g = 0$. This gives the result. □

For the proof of Theorem 1.4, we use the characteristic variational formula for the Bakry–Émery Laplacian (see for example [16, Proposition 1] and [18, Proposition 4]).

$$\lambda_k(\Delta_\phi) = \inf_{V_k} \sup_{f\in V_k} \frac{\int_M |\nabla_g f|^2 e^{-\phi} d\mu_g}{\int_M f^2 e^{-\phi} d\mu_g}, \qquad (35)$$

where V_k is a k-dimensional linear subspace of $H^1(M, \mu_\phi)$.

Proof of Theorem 1.4. According to Lemma 2.3 for $k \in \mathbb{N}^*$ we have a family of k capacitors satisfying properties (I) and (II). For every $\varepsilon > 0$, take $f_i \in \mathcal{T}(F_i, G_i)$, $1 \leq i \leq k$, so that

$$\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g \leq \mathrm{cap}_\phi(F_i, G_i) + \varepsilon.$$

Hence, the characteristic variational formula (35) gives

$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \leq \max_i \frac{\mathrm{cap}_\phi(F_i, G_i) + \varepsilon}{\mu_\phi(F_i)}.$$

Having the properties (I) and (II), we get

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\} \left(\frac{V_\phi([g])}{\mu_\phi(M)}\right)^{\frac{2}{m}} + B(m) \left(\frac{k}{\mu_\phi(M)}\right)^{\frac{2}{m}} + \frac{k\varepsilon}{c(m)\mu_\phi(M)}.$$

Letting ε go to zero, we get the desired inequality. □

Appendix A. Buser type upper bound on Bakry-Émery manifolds

Here, we present a direct and simple proof of a weaker version of Corollary 1.5. This idea of proof was used by Buser [3, Satz 7], Cheng [4], Li and Yau [12] in the case of the Laplace–Beltrami operator. It is based on constructing a family of balls as capacitors which shall be the support of test functions. We can successfully apply this idea in the case of the Bakry–Émery Laplace operator.

Theorem A.1 (Buser type upper bound)**.** *Let (M, g, ϕ) be a compact Bakry–Émery manifold with $\mathrm{Ricci}_\phi(M) > -\kappa^2(m-1)$ and $|\nabla_g \phi| \leq \sigma$ for some $\kappa \geq 0$ and $\sigma \geq 0$. There are positive constants $A(m)$ and $B(m)$ such that for every $k \in \mathbb{N}^*$*

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma, \kappa\}^2 + B(m) \left(\frac{k}{\mu_\phi(M)}\right)^{\frac{2}{m}}.$$

To see that the above theorem is weaker than Corollary 1.5, consider the case where $\mathrm{Ricci}_\phi(M, g)$ is nonnegative. Indeed, the upper bound in Theorem A.1 still depends on σ while Corollary 1.5 provides an upper bound which depends only on the dimension.

Proof. Since $\mathrm{Ricci}_\phi(M) > -\kappa^2(m-1)$ and $|\nabla_g \phi| \leq \sigma$, the comparison theorem gives us the following inequalities for every $0 < r \leq \xi = \min\{\frac{1}{\sigma}, \frac{1}{\kappa}\}$ (with $\xi = \infty$ if $\sigma = \kappa = 0$):

$$\frac{\mu_\phi(B(x,r))}{\mu_\phi(B(x,r/2))} \leq e^{\sigma r} \frac{v(m, r, -\kappa^2)}{v(m, r/2, -\kappa^2)} \leq \sup_{r \in (0,\xi)} e^{\sigma r} \frac{v(m, r, -\kappa^2)}{v(m, r/2, -\kappa^2)} =: c_1(m),$$

and

$$\mu_\phi(B(x,r)) \leq e^{\sigma r} v(m, r, -\kappa^2) \leq \sup_{s \in (0,\xi)} e^{\sigma s} v(m, s, -\kappa^2) r^m =: c_2(m) r^m.$$

Given $k \in \mathbb{N}^*$ let $\rho(k)$ be the positive number defined by

$$\rho(k) = \sup\{r : \exists p_i, \ldots, p_k \in M \text{ with } d_g(p_i, p_j) > r, \forall i \neq j\}.$$

We consider two alternatives:

Case 1. Let $\rho(k) \geq \xi$. For every $r < \xi$, there are k points $p_1, \ldots, p_k$ with $B(p_i, r/2) \cap B(p_j, r/2) = \emptyset$, $\forall i \neq j$. For each $i \in \{1, \ldots, k\}$, we consider

a plateau functions $f_i \in \mathcal{T}(B(p_i, r/4), B(p_i, r/2))$, $1 \le i \le k$, defined as in (19). Then, for every $1 \le i \le k$ and every $r < \xi$

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \le \frac{16}{r^2} \frac{\mu_\phi(B(p_i, r/2))}{\mu_\phi(B(p_i, r/4))} \le c_1(m) \frac{16}{r^2}.$$

Therefore, letting r tend to ξ, one has

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \le c_1(m) \frac{16}{\xi^2} \le A(m) \max\{\sigma, \kappa\}^2.$$

Case 2. Let $\rho(k) < \xi$. Take $r < \rho(k)$ very close to $\rho(k)$. As in Case 1, there are k points $p_1, \dots, p_k$ with $B(p_i, r/2) \cap B(p_j, r/2) = \emptyset$, $\forall i \ne j$. Repeating the same argument we get for every $1 \le i \le k$

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \le c_1(m) \frac{16}{r^2}.$$

Therefore, for every $1 \le i \le k$

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \le c_1(m) \frac{16}{\rho(k)^2}.$$

We now estimate $\rho(k)$. Let $\rho(k) < s < \xi$ and n be the maximal number of points $q_1, \dots, q_n \in M$ so that $d(q_i, q_j) > s$, $\forall i \ne j$. Of course $n \le k$ and because of the maximality of n, the balls $\{B(q_i, s)\}_{i=1}^n$ cover M. Hence, according to Inequality (36)

$$\mu_\phi(M) \le \sum_{i=1}^n \mu_\phi(B(q_i, s)) \le n c_2(m) s^m \le k c_2(m) s^m.$$

Thus, letting s tend to $\rho(k)$ we get

$$\frac{1}{\rho(k)^2} \le c_2(m)^{\frac{2}{m}} \left(\frac{k}{\mu_\phi(M)} \right)^{\frac{2}{m}}.$$

Therefore,

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \le 16 c_1(m) c_2(m)^{\frac{2}{m}} \left(\frac{k}{\mu_\phi(M)} \right)^{\frac{2}{m}}.$$

In conclusion, we obtain

$$\lambda_k(\Delta_\phi) \le \max_i \frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \le A(m) \max\{\sigma, \kappa\}^2 + B(m) \left(\frac{k}{\mu_\phi(M)} \right)^{\frac{2}{m}}.$$

This completes the proof. $\square$

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Université de Neuchâtel, Institut de Mathématiques, Rue Emile-Argand 11, Case postale 158, 2009 Neuchâtel, Switzerland
E-mail address: asma.hassannezhad@unine.ch